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STAVROS VAKEROUDIS* AND MARC YOR†‡

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Abstract

We present a new proof of the extended arc-sine law related to Walsh's Brownian motion, known also as Brownian spider. The main argument mimics the scaling property used previously, in particular by D. Williams [12], in the 1-dimensional Brownian case, which can be generalized to the multivariate case. A discussion concerning the time spent positive by a skew Bessel process is also presented.

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1 Introduction

a) Recently, some renewed interest has been shown (see e.g. [9]) in the study of the law of the vector

$$\overrightarrow{A_1} = \left(\int_0^1 1_{(W_s \in I_i)} ds;\ i = 1, 2, \ldots, n \right) \ ,$$

where (W_s) denotes a Walsh Brownian motion, also called Brownian spider (see [10] for Walsh's lyrical description) living on $I = \bigcup_{i=1}^{n} I_i$, the union of n half-lines of the plane, meeting at 0.

For the sake of simplicity, we assume $p_1 = p_2 = \ldots = p_n = 1/n$, i.e.: when returning to 0, Walsh's Brownian motion chooses, loosely speaking, its "new" ray in a uniform way.

*Department of Mathematics - Probability and Actuarial Sciences group, Université Libre de Bruxelles (ULB), CP210, Boulevard du Triomphe, B-1050 Brussels, Belgium. E-mail: stavros.vakeroudis@ulb.ac.be

†Laboratoire de Probabilités et Modèles Aléatoires (LPMA) CNRS : UMR7599, Université Pierre et Marie Curie - Paris VI, Université Paris-Diderot - Paris VII, 4 Place Jussieu, 75252 Paris Cedex 05, France. E-mail: yormarc@aol.com

‡Institut Universitaire de France, Paris, France.

In fact, excursion theory and/or the computation of the semi-group of Walsh's Brownian motion (see [1]) allow to define the process rigorously.

Since $(d(0, W_s); s \geq 0)$, for d the Euclidian distance, is a reflecting Brownian motion, we denote by $(L_t, t \geq 0)$ the unique continuous increasing process such that: $(d(0, W_s) - L_s; s \geq 0)$ is a $\mathcal{W}_s = \sigma\{W_u, u \leq s\}$ Brownian motion.
Let

$$\overrightarrow{A_t} = \left(A_t^{(1)}, A_t^{(2)}, \ldots, A_t^{(n)}\right)$$

denote the random vector of the times spent in the different rays. In Section 2 we will state and prove our main Theorem concerning the distribution of $\overrightarrow{A_t}$ for a fixed time. Section 3 deals with the general case of stable variables, First, we recall some known results and then we state and prove our main Theorem. Finally, Section 4 is devoted to some remarks and comments.

b) **Reminder on the arc-sine law:**
A random variable A follows the arc-sine law if it admits the density:

$$\frac{1}{\pi\sqrt{x(1-x)}}\ 1_{[0,1)}(x). \tag{1}$$

Some well known representations of an arc-sine variable are the following:

$$A \overset{(law)}{=} \frac{N^2}{N^2 + \hat{N}^2} \overset{(law)}{=} \cos^2(U) \overset{(law)}{=} \frac{T}{T + \hat{T}} \overset{(law)}{=} \frac{1}{1 + C^2}, \tag{2}$$

where $N, \hat{N} \sim \mathcal{N}(0, 1)$ and are independent, U is uniform on $[0, 2\pi]$, T and $\hat{T}$ stand for two iid stable $(1/2)$ unilateral variables, and C is a standard Cauchy variable.
With $(B_t, t \geq 0)$ denoting a real Brownian motion, two well known examples of arc-sine distributed variables are:

$$g_1 = \sup\{t < 1 : B_t = 0\}, \ \text{ and } \ A_1^+ = \int_0^1 ds\ 1_{(B_s > 0)}\ ,$$

a result that is due to Paul Lévy (see e.g. [6, 7, 13]).

c) This point gives some motivation for Section 3. From (2), one could think that more general studies of the time spent positive by diffusions may bring 2 independent gamma variables (this because N^2 and $\hat{N}^2$ are distributed like two independent gamma variables of parameter $1/2$), or 2 independent stable (μ) variables. It turns out that it is the second case which seems to occur more naturally. We devote Section 3 to this case.

2 Main result

Our aim is to prove the following:

Theorem 2.1. *The random vectors* $\overrightarrow{A_T}/T$ *for:*

(i) $T = t$; *(ii)* $T = \alpha_s^{(j)} = \inf\{t : A_t^{(j)} > s\}$; *(iii)* $T = \tau_l$, *the inverse local times,*

have the same distribution. In particular, it is specified by the iid stable (1/2) subordinators:

$$\left(\left(A^{(j)}_{\tau_l}, l \geq 0\right); 1 \leq j \leq n\right).$$

Hence:

$$\overrightarrow{A_1} \overset{(law)}{=} \frac{\overrightarrow{A_{\tau_1}}}{\tau_1}\,, \tag{3}$$

which yields that:

$$\overrightarrow{A_1} \overset{(law)}{=} \left(\frac{T_j}{\sum_{i=1}^{n} T_i}\ ;\ j \leq n\right)\,, \tag{4}$$

where T_j are iid, stable (1/2) variables.

The law of the right-hand side of (3) is easily computed, and consequently so is its left-hand side. We refer the reader to [2] for explicit expressions of this law, which for $n = 2$ reduces to the classical arc-sine law.

Proof of Theorem 2.1.
$a)$ Clearly, (ii) plays a kind of "bridge" between (i) and (iii).
$b)$ We shall work with $\left(\alpha^{(1)}_s, s \geq 0\right)$, the inverse of $\left(A^{(1)}_t, t \geq 0\right)$. It is more convenient to use the notation $\left(\alpha^{(+)}_s, s \geq 0\right)$ for $\left(\alpha^{(1)}_s, s \geq 0\right)$. We then follow the main steps of [13] (Section 3.4, p. 42), which themselves are inspired by Williams [12]; see also Watanabe (Proposition 1 in [11]) and Mc Kean [8].
$\left(A^{(j)}_t\right)$ denotes the time spent in I_j, for any $j \neq 1$. Since

$$\begin{cases} A^{(j)}_{\alpha^{(+)}_1} = A^{(j)}_{\tau(L_{\alpha^{(+)}_1})} \overset{(law)}{=} (L_{\alpha^{(+)}_1})^2 A^{(j)}_{\tau_1}\,, \\ \alpha^{(+)}_1 = 1 + \sum_j A^{(j)}_{\alpha^{(+)}_1}\,, \\ \quad \text{and} \\ \text{for every } \ u, t \geq 0, \quad \left(L^2_{\alpha^{(+)}_u} < t\right) = \left(u < A^{(1)}_{\tau_{\sqrt{t}}}\right)\,, \end{cases}$$

and invoking the scaling property, we can write jointly for all j's:

$$\begin{aligned} \left(A^{(j)}_{\alpha^{(+)}_1}, L^2_{\alpha^{(+)}_1}, \alpha^{(+)}_1\right) &\overset{(law)}{=} \left(L^2_{\alpha^{(+)}_1} A^{(j)}_{\tau_1}, L^2_{\alpha^{(+)}_1}, 1 + \sum_j L^2_{\alpha^{(+)}_1} A^{(j)}_{\tau_1}\right) \\ &\overset{(law)}{=} \left(\frac{A^{(j)}_{\tau_1}}{A^{(1)}_{\tau_1}}, \frac{1}{A^{(1)}_{\tau_1}}, \frac{\tau_1}{A^{(1)}_{\tau_1}}\right). \end{aligned} \tag{5}$$

Dividing now both sides by $\alpha^{(+)}_1$ and remarking that: $\alpha^{(+)}_1 A^{(1)}_{\tau_1} = \tau_1$, we deduce:

$$\frac{1}{\alpha^{(+)}_1}\left(A^{(j)}_{\alpha^{(+)}_1}, L^2_{\alpha^{(+)}_1}\right) \overset{(law)}{=} \frac{1}{\tau_1}\left(A^{(j)}_{\tau_1}, 1\right). \tag{6}$$

With the help of the scaling Lemma below, we obtain:

$$\begin{aligned}
E\left[1_{(W_1\in I_1)}f(\overrightarrow{A_1},L_1^2)\right] &= E\left[\frac{1}{\alpha_1^{(+)}}f\left(\frac{\overrightarrow{A_{\alpha_1^{(+)}}}}{\alpha_1^{(+)}},\frac{L^2_{\alpha_1^{(+)}}}{\alpha_1^{(+)}}\right)\right]\\
&\overset{\text{from (5)}}{=} E\left[\frac{A^{(1)}_{\tau_1}}{\tau_1}f\left(\frac{\overrightarrow{A_{\tau_1}}}{\tau_1},\frac{1}{\tau_1}\right)\right]. \qquad (7)
\end{aligned}$$

I_1 may be replaced by I_m, for any $m\in\{2,\dots,n\}$. Adding the m quantities found in (7) and remarking that:

$$\tau_1=\sum_{i=1}^{n}A^{(i)}_{\tau_1}\ , \qquad (8)$$

we get:

$$E\left[f(\overrightarrow{A_1},L_1^2)\right] \ = \ E\left[f\left(\frac{\overrightarrow{A_{\tau_1}}}{\tau_1},\frac{1}{\tau_1}\right)\right].$$

which proves (3). Note that from (6), the latter also equals:

$$E\left[f\left(\frac{\overrightarrow{A_{\alpha_1^{(+)}}}}{\alpha_1^{(+)}},\frac{L^2_{\alpha_1^{(+)}}}{\alpha_1^{(+)}}\right)\right].$$

Equality in law (4) follows now easily. Indeed, we denote by $\boldsymbol{\nu}$ the Itô measure of the Brownian spider, and we have:

$$\boldsymbol{\nu}=\frac{1}{n}\sum_{j=1}^{n}\nu_j\ , \qquad (9)$$

where ν_j is the canonical image of $\mathbf{n}$, the standard Itô measure of the space of the excursions of the standard Brownian motion, on the space of the excursions on I_j. Hence, with $\lambda_j,\ j=1,\dots,n$ denoting positive constants:

$$\begin{aligned}
E\left[\exp\left(-\sum_{j=1}^{n}\lambda_j A^{(j)}_{\tau_1}\right)\right] &= \exp\left(-\frac{1}{n}\sum_{j=1}^{n}\int\nu_j(d\varepsilon_j)(1-e^{-\lambda_j\nu_j})\right)\\
&= \exp\left(-\frac{1}{n}\sum_{j=1}^{n}\sqrt{2\lambda_j}\right)\ ,
\end{aligned}$$

thus:

$$\overrightarrow{A_{\tau_1}}=\left(A^{(j)}_{\tau_1}\ ;\ j\le n\right)\overset{(law)}{=}\left(\frac{1}{n^2}T_j\ ;\ j\le n\right).$$

The latter, using (8) yields:

$$\overrightarrow{A_1}=\frac{\overrightarrow{A_{\tau_1}}}{\tau_1}=\frac{\overrightarrow{A_{\tau_1}}}{\sum_{i=1}^{n}A^{(i)}_{\tau_1}}\overset{(law)}{=}\left(\frac{T_j}{n^2\sum_{i=1}^{n}n^{-2}T_i}\ ;j\le n\right),$$

which finishes the proof. ∎

It now remains to state the scaling Lemma which played a role in (7), and which we lift from [13] (Corollary 1, p. 40) in a "reduced" form.

Lemma 2.2. (Scaling Lemma) *Let $U_t = \int_0^t ds\theta_s$, with the pair (W, θ) satisfying:*

$$(W_{ct}, \theta_{ct}; t \geq 0) \stackrel{(law)}{=} \left(\sqrt{c}W_t, \theta_t; t \geq 0\right) . \tag{10}$$

Then,

$$E\left[F\left(W_u, u \leq 1\right)\theta_1\right] = E\left[\frac{1}{\alpha_1}F\left(\frac{1}{\sqrt{\alpha_1}}W_{v\alpha_1}, v \leq 1\right)\right] , \tag{11}$$

where $\alpha_t = \inf\{s : U_s > t\}$.

3 Stable subordinators

3.1 Reminder and preliminaries on stable variables

In this Section, we consider S_μ and S'_μ two independent stable variables with exponent $\mu \in (0, 1)$, i.e. for every $\lambda \geq 0$, the Laplace transform of S_μ is given by:

$$E[\exp(-\lambda S_\mu)] = \exp(-\lambda^\mu). \tag{12}$$

Concerning the law of S_μ, there is no simple expression for its density (except for the case $\mu = 1/2$; see e.g. Exercise 4.20 in [3]). However, we have that, for every $s < 1$ (see e.g. [15] or Exercise 4.19 in [3]):

$$E[(S_\mu)^{\mu s}] = \frac{\Gamma(1-s)}{\Gamma(1-\mu s)} . \tag{13}$$

We consider now the random variable of the ratio of two μ-stable variables:

$$X = \frac{S_\mu}{S'_\mu} . \tag{14}$$

Following e.g. Exercise 4.23 in [3], we have respectively the following formulas for the Stieltjes and the Mellin transforms of X:

$$E\left[\frac{1}{1+sX}\right] = \frac{1}{1+s^\mu} \ , \ s \geq 0 \ , \tag{15}$$

$$E\left[X^s\right] = \frac{\sin(\pi s)}{\mu \sin(\frac{\pi s}{\mu})}, \ 0 < s < \mu \ . \tag{16}$$

Moreover, the density of the random variable X^μ is given by (see e.g. [14, 5] or Exercise 4.23 in [3]):

$$P(X^\mu \in dy) = \frac{\sin(\pi\mu)}{\pi\mu} \ \frac{dy}{y^2 + 2y\cos(\pi\mu) + 1} \ , \ y \geq 0, \tag{17}$$

or equivalently:

$$\left(\frac{S_\mu}{S'_\mu}\right)^\mu = (C_\mu | C_\mu > 0), \tag{18}$$

where, with C denoting a standard Cauchy variable and U a uniform variable in $[\,0, 2\pi\,)$,

$$C_\mu = \sin(\pi\mu)C - \cos(\pi\mu) \overset{(law)}{=} \frac{\sin(\pi\mu - U)}{U} \ .$$

3.2 The case of 2 stable variables

We turn now our study to the random variable:

$$A = \frac{S'_\mu}{S'_\mu + S_\mu} = \frac{1}{1+X}, \tag{19}$$

Theorem 3.1. *The density function of the random variable A is given by:*

$$P\left(A \in dz\right) = \frac{\sin(\pi\mu)}{\pi} \frac{dz}{z(1-z)\left[\left(\frac{1-z}{z}\right)^\mu + \left(\frac{z}{1-z}\right)^\mu + 2\cos(\pi\mu)\right]} \ , \quad z \in [0,1]. \tag{20}$$

Proof of Theorem 3.1.
Identity (19) is equivalent to:

$$X = \frac{1}{A} - 1 \ .$$

Hence, (15) yields:

$$E\left[\frac{1}{1+sX}\right] = E\left[\frac{A}{(1-s)A+s}\right] = \frac{1}{1+s^\mu} \ .$$

We consider now a test function f and invoking the density (17) we have ($\nu = \frac{1}{\mu} > 1$):

$$E\left[f\left(\frac{1}{1+X}\right)\right] = \frac{\sin(\pi\mu)}{\pi\mu} \int_0^\infty \frac{dy}{y^2 + 2y\cos(\pi\mu) + 1} \, f\left(\frac{1}{1+y^\nu}\right).$$

Changing the variables $z = \frac{1}{1+y^\nu}$, we deduce:

$$E\left[f\left(A\right)\right] = \frac{\sin(\pi\mu)}{\pi} \int_0^1 \frac{dz(1-z)^{\mu-1}}{z^{\mu+1}} \, f\left(z\right)\Delta(z),$$

where:

$$\begin{aligned}\Delta(z) &= \frac{1}{(z^{-1}-1)^{2\mu} + 2(z^{-1}-1)^\mu\cos(\pi\mu) + 1} \\ &= \frac{z^{2\mu}}{(1-z)^{2\mu} + 2(1-z)^\mu z^\mu \cos(\pi\mu) + z^{2\mu}} \ ,\end{aligned}$$

and (20) follows easily. ∎

In Figure 1, we have plotted the density function g of A, for several values of μ.

Remark 3.2. *Similar discussions have been made in [4] in the framework of a skew Bessel process with dimension $2-2\alpha$ and skewness parameter p. Formula (20) is a particular case of formula in [4] for the density of the time spent positive (called $f_{p,\alpha}$ in [4]).*

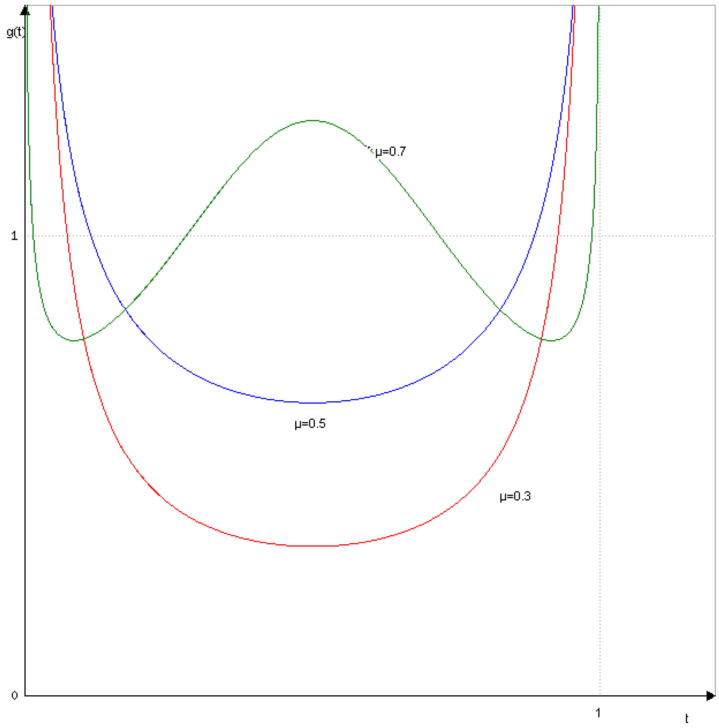

Figure 1: The density function g of A, for several values of μ.

3.3 The case of many stable (1/2) variables

In this Subsection, we consider again n iid stable (1/2) variables, i.e.: $T_1, \dots, T_n$, and we will study the distribution of:

$$A_1^{(1)} = \frac{T_1}{T_1 + \ldots + T_n} \ . \tag{21}$$

The following Theorem answers to an open question (and even in a more general sense) stated at the end of [9].

Theorem 3.3. *The density function of the random variable $A_1^{(1)}$ is given by:*

$$P\left(A_1^{(1)} \in dz\right) = \frac{1}{\pi} \frac{dz}{\sqrt{z}\sqrt{1-z}\left[(n-1)z + \frac{1}{n-1}(1-z)\right]} \ , \quad z \in [0,1]. \tag{22}$$

Proof of Theorem 3.3.
We first remark that, with C denoting a standard Cauchy variable, using e.g. (2):

$$A_1^{(1)} \overset{(law)}{=} \frac{T_1}{T_1 + (n-1)^2 T_2} \overset{(law)}{=} \frac{1}{1 + (n-1)^2 C^2} \ . \tag{23}$$

Hence, with f standing again for a test function, and invoking the density of a standard

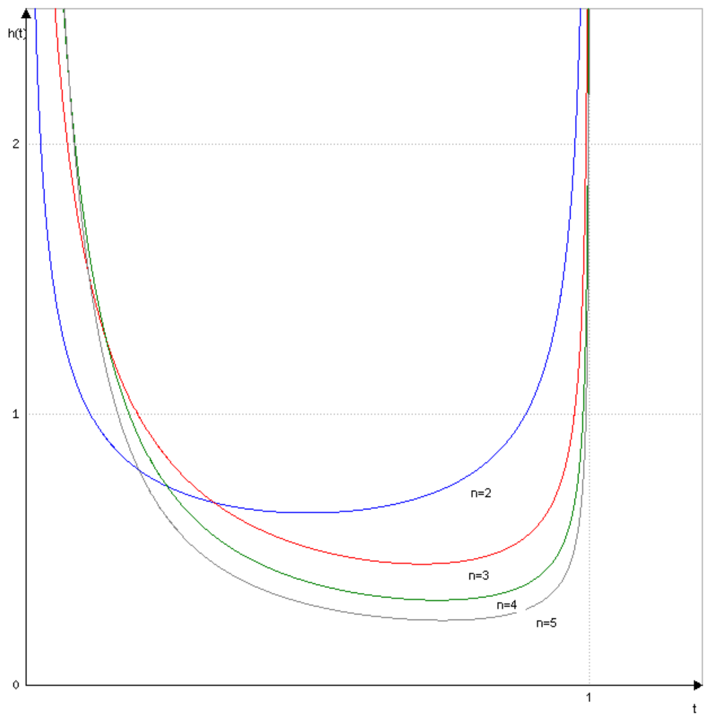

Figure 2: The density function h of $A_1^{(1)}$, for several values of n.

Cauchy variable, that is: for every $x \in \mathbb{R}$, $g(x) = \frac{1}{\pi(1+x^2)}$ we have:

$$\begin{aligned}
E\left[f\left(A_1^{(1)}\right)\right] &= E\left[f\left(\frac{1}{1+(n-1)^2C^2}\right)\right] \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dx}{1+x^2}\, f\left(\frac{1}{1+(n-1)^2x^2}\right) \\
&\overset{x^2=y}{=} \frac{2}{\pi}\int_0^{\infty}\frac{dy}{2\sqrt{y}(1+y)}\, f\left(\frac{1}{1+(n-1)^2y}\right)
\end{aligned}$$

Changing the variables $z = \frac{1}{1+(n-1)^2y}$, we deduce:

$$E\left[f\left(A_1^{(1)}\right)\right] = \frac{1}{\pi}\int_0^1\frac{dz}{(n-1)^2z^2}\,\frac{(n-1)\sqrt{z}}{\sqrt{z-1}\left(1+\frac{1}{(n-1)^2}\left(\frac{1}{z}-1\right)\right)}\, f(z)\,,$$

and (22) follows easily. ∎

Figure 2 presents the plot of the density function h of $A_1^{(1)}$, for several values of n.

Corollary 3.4. *The following convergence in law holds:*

$$n^2A_1^{(1)}(n) \xrightarrow[n\to\infty]{(law)} C^2\ . \tag{24}$$

Proof of Corollary 3.4.
It follows from Theorem 3.3 by simply remarking that $C \overset{(law)}{=} C^{-1}$. Hence:

$$n^2 A_1^{(1)}(n) = \frac{n^2}{1+(n-1)^2 C^2} = \frac{1}{\frac{1}{n^2} + \left(\frac{n-1}{n}\right)^2 C^2} \overset{n\to\infty}{\longrightarrow} \frac{1}{C^2} \overset{(law)}{=} C^2.$$

∎

4 Conclusion and comments

We end up this article with some comments: usually, a scaling argument is "one-dimensional", as it involves a time-change. Exceptionally (or so it seems to the authors), here we could apply a scaling argument in a multivariate framework. We insist that the scaling Lemma plays a key role in our proof. The curious reader should also look at the totally different proof of this Theorem in [2], which mixes excursion theory and the Feynman-Kac method.

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References

[1] M.T. Barlow, J.W. Pitman and M. Yor (1989). On Walsh's Brownian motion. *Sém. Prob. XXIII, Lect. Notes in Math.*, **1372**, Springer, Berlin Heidelberg New York, pp. 275-293.

[2] M.T. Barlow, J.W. Pitman and M. Yor (1989). Une extension multidimensionnelle de la loi de l'arc sinus. *Sém. Prob. XXIII, Lect. Notes in Math.*, **1372**, Springer, Berlin Heidelberg New York, pp. 294-314.

[3] L. Chaumont and M. Yor (2012). Exercises in Probability: A Guided Tour from Measure Theory to Random Processes, via Conditioning. Cambridge University Press, 2nd Edition.

[4] Y. Kasahara and Y. Yano (2005). On a generalized arc-sine law for onedimensional diffusion processes. *Osaka J. Math.*, **42**, pp. 1-10.

[5] J. Lamperti (1958). An occupation time theorem for a class of stochastic processes. *Trans. Amer. Math. Soc.*, **88**, pp. 380-387.

[6] P. Lévy (1939). Sur un problème de M. Marcinkiewicz. *C.R.A.S.*, **208**, pp. 318-321. Errata p. 776.

[7] P. Lévy (1939). Sur certains processus stochastiques homogènes. *Compositio Math.*, t. **7**, pp. 283-339.

[8] H.P. McKean (1975). Brownian local time. *Adv. Math.*, **16**, pp. 91-111.

[9] V.G. Papanicolaou, E.G. Papageorgiou and D.C. Lepipas (2012). Random Motion on Simple Graphs. *Methodol. Comput. Appl. Probab.*, **14**, pp. 285-297.

[10] J.B. Walsh (1978). A diffusion with discontinuous local time. *Astérisque*, **52-53**, pp. 37-45.

[11] S. Watanabe (1995). Generalized arc-sine laws for one-dimensional diffusion processes and random walks. *Proc. Sympos. Pure Math.*, **57**, Stoch. Analysis, Cornell University (1993), Amer. Math. Soc., pp. 157-172.

[12] D. Williams (1969). Markov properties of Brownian local time. *Bul. Am. Math. Soc.*, **76**, pp. 1035-1036.

[13] M. Yor (1995). Local times and Excursions for Brownian motion: a concise introduction. Lecciones en Mathematicas Universidad Central de Venezuela, Caracas.

[14] V.M. Zolotarev (1957). Mellin-Stieltjes transforms in probability theory. *Teor. Veroyatnost. i Primenen.*, **2**, pp. 444-469.

[15] V.M. Zolotarev (1994). On the representation of the densities of stable laws by special functions. (In Russian.) *Teor. Veroyatnost. i Primenen.*, **39**, no. 2, pp. 429-437; translation in *Theory Probab. Appl.*, (1995) **39**, no. 2, pp. 354-362.